# Solving quadratic assignment problem using linear reformulation model

Wuttinan Nunkaew[1)] and Busaba Phruksaphanrat,[2)]

[1), 2)]Department of Industrial Engineering, Thammasat University, Pathumthani 12121, Thailand

## Abstract

This research is a contribution on allocation of *n* facilities to *n* locations in a Quadratic Assignment Problem (QAP). The conventional QAP model immerses in minimization of flow-distance objective function coined in quadratic form. It is such a complicated nonlinear equation that general linear solver package cannot generate the optimal solutions. Although linearization techniques can be applied to transform the quadratic term to be a linearized version of the QAP, a number of additional variables and constraints also have to be afforded. In this research, we develop a new linear reformulation function which is applied to the proposed Linear Reformulation Model (LRM) for the QAP. Using the proposed LRM, the Decision Makers (DMs) do not have to solve the QAP by special solver package. A simple linear solver can be used to determine the optimal solutions of the QAP. Furthermore, the drawbacks of reformulating the QAP by linearization with supplementary variables and constraints can be efficiently terminated

**Keywords:** Quadratic assignment problem, Facility layout, Linear reformulation model

## 1. Introduction

A Quadratic Assignment Problem (QAP) originally introduced by Koopmans and Beckmann in 1957 [1] concerns with the objective function depending on combinations of decisions. It has been applied to a variety of practical applications including campus planning [2], design of typewriter keyboards [3, 4], data allocation [5], control panel design [6], container re-handling operations [7] and many others. One of the most common cases resulting in QAP is facility location or facility layout problem [8, 9, 10, 11], dealing with assignment of n facilities to n locations. The objective function of QAP involves interaction costs between pairs of facilities [8, 12]. The impact of one decision cannot be evaluated until others are determined [9]. So, this kind of assignment problem cannot be fitted to the linear objective function.

QAP is categorized in the class of the most complicated NP-hard combinatorial optimization problems [8, 13, 14]. Solving large QAP instances within a reasonable time seems to be impossible due to the computational limits. In general, exact solutions can be generated only for QAP with size of n less than or equal to 30 [8, 15, 16]. Nevertheless, substantial power of computation is also required.

To enhance the applicable ways of solving the QAP, several linearization algorithms and other techniques have been introduced. Wu et al. [17] presented a global optimization method for QAP. Some numerical instances were provided to show the efficiency of the given method. Mixed-Integer Linear Programming (MILP) for the QAP, based on the splitting of the coefficient matrices, were introduced by Wright [15]. Nyberg and Westerlund [18] also presented solving the QAP by MILP with discrete linear reformulation. Zhang et al. [19] formulated QAP-MILP extended from Kaufman and Broeckx formulation [20]. Although, these MILP have given easier computational manners than the conventional QAP, some additional variables and constraints were needed.

A number of solution procedures, including Reformulation Linearization Techniques (RLT) [8, 21, 22, 23], Parallel Hybrid Algorithm (PHA) [24], and Memetic Algorithm (MA) [25] were also presented. These solution methods performed very well on their large-scale instances, however, they provided the complicated algorithms and were quite difficult in applying to the practical QAP.

The point that we have noticed is not only the performance of solving the QAP, involving how big of the instances' size, but the easiness in applying the method to the QAP should be significantly concerned. In fact, especially in the practical facility layout problems, the DMs do not have to deal with such very large values of n in the QAP or he/she can decompose the large problem into a group of problems. Conversely, they all need easy-to-apply procedure to solve this problem.

For the proposed study, we have developed and introduced a Linear Reformulation Model (LRM) with linear max function for solving the QAP instead of the conventional quadratic model. The key idea is to transform the quadratic objective function to a linear form. The max function has been applied to maintain the value of binary combination function.

## 2. Quadratic assignment problem

Consider the conventional QAP introduced by Koopmans and Beckmann [1] with minimization of a quadratic objective function, the general model can be expressed as follows [9]:

**QAP:**

$$\min f(x_{ij}, x_{kl}) = \sum_{i}^{n}\sum_{j}^{n}\sum_{k>i}^{n}\sum_{l\neq j}^{n} f_{ik} d_{jl} x_{ij} x_{kl} \tag{1}$$

subject to

$$\sum_{j}^{n} x_{ij} = 1, \text{ for all } i, \tag{2}$$

$$\sum_{i}^{n} x_{ij} = 1, \text{ for all } j, \tag{3}$$

$$x_{ij} = 0 \text{ or } 1, \text{ for all } i \text{ and } j, \tag{4}$$

where $f_{ik}$ is the flow matrix and $d_{jl}$ is the distance matrix.

Equation (1) is the objective function of total cost concerning the amount of flow between the facilities and their distances between locations. It is minimized to allocate each facility to a location. Constraint (2) forces that only one facility can be assigned to one location. Constraint (3) ensures that each location can be assigned to only one facility. The binary variables are defined in (4).

More general expression of QAP was presented by Lawler [26]. A four-dimensional cost coefficients (see [23]), $c_{ijkl}$ , were introduced to replace two matrices of $f_{ik}$ and $d_{jl}$ . Involving two assignment decisions, $c_{ijkl}$ is applied only if $i$ is assigned to $j$, and $k$ is assigned to $l$, that is, $x_{ij} = 1$ and $x_{kl} = 1$.

## 3. General linearization of a quadratic function

In general, a quadratic function can be linearized as introduced by Chen et al. [27]. To transform quadratic term into a linear binary programming, new additional binary variables, $w_{ijkl}$, have to be used to replace a product of $x_{ij}x_{kl}$ with three additional constraints as follows:

**L-QAP:**

$$\min g(x_{ij}, x_{kl}) = \sum_{i}^{n}\sum_{j}^{n}\sum_{k>i}^{n}\sum_{l\neq j}^{n} f_{ik} d_{jl} w_{ijkl} \tag{5}$$

subject to

$$x_{ij} + x_{kl} \leq w_{ijkl} + 1, \text{ for all } i, j, k \text{ and } l, \tag{6}$$

$$x_{ij} + x_{kl} \geq 2w_{ijkl}, \text{ for all } i, j, k \text{ and } l, \tag{7}$$

$$w_{ijkl} = 0 \text{ or } 1, \text{ for all } i, j, k \text{ and } l, \tag{8}$$

and (2) - (4).

Equation (5) is the linearized objective function such that term $x_{ij}x_{kl}$ in (1) is replaced by $w_{ijkl}$. Its values correspond to the values of produce of $x_{ij}x_{kl}$. Constraints (6) and (7) are the upper and lower bound of $x_{ij} + x_{kl}$, consequently. Defining the binary property of $w_{ijkl}$ is expressed in (8).

The previous L-QAP model employs linearization to reduce the complexity of the quadratic function, nevertheless, it also needs non-negligible additional variables and constraints. This means that higher-variable spaces in computation have to be served.

## 4. Linear reformulation model

The combinations of $x_{ij}$ and $x_{kl}$ in a quadratic function are summarized in Table 1. The product term of $x_{ij}x_{kl}$ in objective function (1) gives two possible solutions, which are

$$x_{ij}x_{kl} = \begin{cases} 1, \text{ if } x_{ij} \text{ and } x_{kl} = 1 \\ 0, \text{ if } x_{ij} \text{ or } x_{kl} = 0 \end{cases}, \tag{9}$$

**Table 1** Linear reformulation function

| Combination (A) | | $x_{ij}x_{kl}$ (B) | $x_{ij}+x_{kl}-1$ (C) | max $[0, x_{ij}+x_{kl}-1]$ (D) |
|---|---|---|---|---|
| $x_{ij}$ | $x_{kl}$ | | | |
| 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | -1 | 0 |

To avoid the difficulty of quadratic function, we develop a linear reformulation function in the form of linear max function [28, 29, 30] presented in (10) to replace the product term of $x_{ij}x_{kl}$. With the proposed linear reformulation function, the Linear Reformulation Model (LRM) for QAP can be formulated and explained in the following theorem.

**LRM:**

$$\min\ h(x_{ij}, x_{kl}) = \sum_{i}^{n}\sum_{j}^{n}\sum_{k>i}^{n}\sum_{l\neq j}^{n} f_{ik}d_{jl}\ \max\,[0, x_{ij}+x_{kl}-1], \tag{10}$$

subject to

(2) - (4).

**Theorem 1**

*For any quadratic assignment problem, QAP model is equivalent to LRM with objective function (10) and constraints (2) – (4) in the sense that they have the same optimal solutions.*

**Proof**

Let $f^{"}(x_{ij}, x_{kl}) = x_{ij}x_{kl}$ and $h^{"}(x_{ij}, x_{kl}) = x_{ij}+x_{kl}-1$ be the combination functions of binary variables $x_{ij}$ and $x_{kl}$. For any combination of $x_{ij}$ and $x_{kj}$ in column (A) of Table 1, two possible cases are concerned. Both $f^{"}(x_{ij}, x_{kl})$ and $h^{"}(x_{ij}, x_{kl})$ obtain the value of 1 if and only if $x_{ij} = 1$ and $x_{kl} = 1$. Otherwise, $f^{"}(x_{ij}, x_{kl})$ and $h^{"}(x_{ij}, x_{kl})$ are neither greater nor equal to 1 as shown in columns (B) and (C), respectively. With a max function in column (D), $h^{"}(x_{ij}, x_{kl})$ will be forced to take the value of 1 if $x_{ij}$ and $x_{kl}$ are both equal to 1. Otherwise, $h^{"}(x_{ij}, x_{kl})$ will be forced by max function to take the value of 0 if any one of $x_{ij}$ and $x_{kl}$ is equal to 0. So, $h^{"}(x_{ij}, x_{kl})$ with max function gives the same optimal solutions as those of $f^{"}(x_{ij}, x_{kl})$ in column (B). Then, we have

$$\max\,[0, x_{ij}+x_{kl}-1] = \begin{cases} 1, \text{ if } x_{ij} \text{ and } x_{kl} = 1, \\ 0, \text{ if } x_{ij} \text{ or } x_{kl} = 0 \end{cases} \tag{11}$$

which is the same solution as (9), obviously.

## 5. Illustrative example, evaluation and discussion

To demonstrate the capability of the proposed LRM for QAP, the facility layout problem presented in [9] is selected to be an illustrative example. Mall managers want to arrange four stores in the four locations (n = 4) to minimize customer inconvenience. So, they need to decide which unit to assign to each location.

The four expected tenants (Clothes Are, Computers Aye, Toy Parade, and Book Bazaar) for the shop locations are summarized in Table 2. The number of customers, who might wish to visit various pair of shops each week (in thousands), is also collected in the table. For example, expected 5 thousand customers per week will visit both shop 1 (Clothes Are) and shop 2 (Computers Aye). The adjacent walking distances (in feet) between the shop locations are listed in Table 3. For instance, the distance between locations 1 and 4 is a hundred and seventy feet.

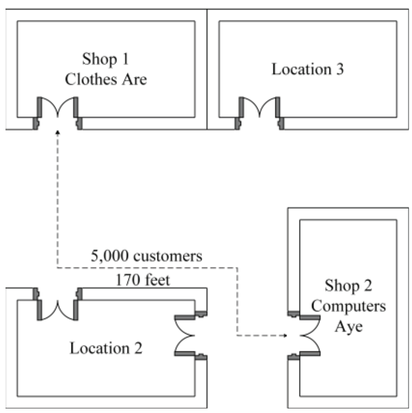


**Figure 1** Example flow-distance of shops 1 and 4

**Table 2** Mall layout example tenants

| | | Common Customers | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Store, $i$ | 1: Clothes Are | _ | 5 | 2 | 7 |
| | 2: Computers Aye | 5 | _ | 3 | 8 |
| | 3: Toy Parade | 2 | 3 | _ | 3 |
| | 4: Book Bazaar | 7 | 8 | 3 | _ |

**Table 3** Walking distances

| | | To $l$ | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| From $j$ | 1 | _ | 80 | 150 | 170 |
| | 2 | 80 | _ | 130 | 100 |
| | 3 | 150 | 130 | _ | 120 |
| | 4 | 170 | 100 | 120 | _ |

The flow-distance measure is the product of flow volumes between shops and the distances between their assigned locations. For example, if shop 1 (Clothes Are) is assigned to location 1 and shop 2 (Computers Aye) is located at space 4 as shown in Figure 1, their 5 thousand expected customers will have to walk 170 feet between the locations. Then, this increases 5 × 170 or 850 thousand customer-feet to the flow-distance.

The formulation of the conventional QAP, the L-QAP and the proposed LRM models for the mall layout problem can be shown as follows:

**QAP:**

$$\min\ f(x_{ij}, x_{kl}) = \sum_{i}^{4}\sum_{j}^{4}\sum_{k>i}^{4}\sum_{l\neq j}^{4} f_{ik} d_{jl} x_{ij} x_{kl} \qquad (12)$$

subject to

$$x_{11} + x_{12} + x_{13} + x_{14} = 1, \qquad (13)$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 1, \qquad (14)$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 1, \qquad (15)$$

$$x_{41} + x_{42} + x_{43} + x_{44} = 1, \qquad (16)$$

$$x_{11} + x_{21} + x_{31} + x_{41} = 1, \qquad (17)$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 1, \qquad (18)$$

$$x_{13} + x_{23} + x_{33} + x_{43} = 1, \qquad (19)$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 1, \qquad (20)$$

$$x_{ij} = 0 \text{ or } 1,\ i \text{ and } j = 1,\ldots,4. \qquad (21)$$

**L-QAP:**

$$\min\ g(x_{ij}, x_{kl}) = \sum_{i}^{4}\sum_{j}^{4}\sum_{k>i}^{4}\sum_{l\neq j}^{4} f_{ik} d_{jl} w_{ijkl} \qquad (22)$$

subject to

$$x_{ij} + x_{kl} \leq w_{ijkl} + 1, \qquad (23)$$

$$x_{ij} + x_{kl} \geq 2w_{ijkl}, \qquad (24)$$

$$w_{ijkl} = 0 \text{ or } 1, \qquad (25)$$

and (13) - (21).

where $i$, $j$, $k$ and $l$ = 1,…, 4.

**LRM:**

$$\min\ h(x_{ij}, x_{kl}) = \sum_{i}^{4}\sum_{j}^{4}\sum_{k>i}^{4}\sum_{l\neq j}^{4} f_{ik} d_{jl}\ \max\,[0, x_{ij} + x_{kl} - 1] \qquad (26)$$

subject to

(13) - (21).

All three models are solved by Premium Solver Platform V10.5.0.0 (which can be cooperative working with spreadsheets in Microsoft® Excel) on a PC with Intel® Core™ i7-6500U @2.50 GHz CPU and 8 GB RAM.

The optimal solutions obtained from the conventional QAP, the L-QAP and the proposed LRM for QAP place shop 1 (Clothes Are) in location 1, shop 2 (Computers Aye) in location 4, shop 3 (Toy Parade) in location 3, and shop 4 (Book Bazaar) in location 2. The total flow-distance of 3,260 thousand customer-feet is generated.

For the conventional QAP model and the proposed LRM for QAP, they give the same decision variables of $x_{ij}$ and $x_{kl}$. That are $x_{11} = x_{24} = x_{33} = x_{42} = 1$, and other $x_{ij} = 0$. Whereas, the L-QAP model also requires additional decision variables of $w_{ijkl}$. That are $w_{1124} = w_{1133} = w_{1142} = w_{2433} = w_{2442} = w_{3342} = 1$, and other $w_{ijkl} = 0$. The optimal total flow-distance can be calculated as (27).

$$5(170) + 2(150) + 7(80) + 3(120) + 8(100) + 3(130) = 3,260 \quad (27)$$

As demonstrated in previous QAP, L-QAP and LRM models, the QAP model is totally nonlinear so that it might cause difficulty in solving this problem by general non-commercial solver. L-QAP and LRM are linear model, however, L-QAP needs additional variables and constraints regarding number of *i, j, k* and *l* as shown in Table 4.

Moreover, the proposed LRM is able to solve by non-commercial linear solver and more practical comparing to the L-QAP.

**Table 4** Comparisons of QAP, L-QAP and the proposed LRM

| Content | QAP | L-QAP | Proposed LRM |
|---|---|---|---|
| Linearity | Non | Linear | Linear |
| Additional variables | - | ($i$×$j$×$k$×$l$) | - |
| Additional constraints | - | 2($i$×$j$+$k$×$l$) | - |
| Easiness to compute | ✗ | ✗ | ✓ |
| Applying linear solver | ✗ | ✓ | ✓ |

## 6. Conclusion

In this paper, we propose the new LRM for the QAP. A linear reformulation function was formulated in the form of max function and applied to the objective function of LRM. This makes the convention QAP model transformed to a linear model with the objective function of minimizing the flow-distance. In order to demonstrate the capability of the proposed LRM, three QAP models (the conventional QAP model, the L-QAP model, and the proposed LRM for QAP) are applied to solve a shopping mall layout problem. Comparing to the conventional QAP, the proposed LRM gives the same optimal solutions, however, a special solver package for quadratic function is needless using the proposed LRM. By simple existing linear solver, the proposed LRM can obtain the optimal solutions. Although the L-QAP model employing the linearization technique also obtain the identical solutions for the example, it requires a number of additional variables and constraints. This creates more complexity for the DMs' formulation. Using the L-QAP model, he/she has to spend more efforts to devise the linearized objective function of the QAP, which can sometimes make an error. The proposed LRM outperforms both conventional QAP and L-QAP models in the features of linear property and easy-to-apply method.

## 7. Acknowledgements

This research was supported by the Faculty of Engineering, Thammasat University, Thailand.

## 8. References

[1] Koopmans, T. C., & Beckmann, M. (1957). Assignment problems and the location of economic activities. Econometrica, 25(1), 53-76.

[2] Dickey, J. W., & Hopkins, J. W. (1972). Campus building arrangement using topaz. Transportation Research, 6(1), 59-68.

[3] Burkard, R. E., & Offermann, J. (1977). Entwurf von Schreibmaschinentastaturen mittels quadratischer Zuordnungsprobleme. Zeitschrift für Operations Research, 21(4), B121–B132.

[4] Amaral, A. R. S. (2013). A parallel ordering problem in facilities layout. Computers & Operations Research, 40, 2930–2939.

[5] Adl, R. K., & Rankoohi, S. M. T. R. (2009). A New Ant Colony Optimization Based Algorithm for Data Allocation Problem in Distributed Databases. Knowledge and Information Systems, 20, 349-373.

[6] Pollatschek, M., Gershoni, N., & Radday, Y. (1976). Optimization of the typewriter keyboard by simulation. Angewandte Informatik, 17, 438-439.

[7] Cordeau, J. F., Gaudioso, M., Laporte, G., & Moccia, L. (2007). The service allocation problem at the Gioia Tauro Maritime Terminal. European Journal of Operational Research, 176, 1167–1184.

[8] Rostami, B., & Malucelli, F. (2014). A revised reformulation-linearization technique for the quadratic assignment problem. Discrete Optimization, 14, 97-103.

[9] Rardin, R. L. (1998). Optimization in Operations Research. USA: Prentice-Hall.

[10] Tompkins, J. A., White, J. A., Bozer, Y. A., & Tanchoco, J. M. A. (2003). Facilities Planning. USA: John Wiley & Sons.

[11] Hahn, P. M., & Krarup, J. (2001). A hospital facility layout problem finally solved. Journal of Intelligent Manufacturing, 12, 487–496.

[12] Padberg, M., & Rijal, M. P. (1996). Location, Scheduling, Design and Integer Programming. Boston: Kluwer.

[13] Sahni, S., & Gonzalez, T. (1976). P-Complete Approximation Problems. Journal of the ACM, 23, 555–565.

[14] Frieze, A. M., & Yadegar, J. (1983). On the quadratic assignment problem. Discrete Applied Mathematics, 5(1), 89-98.

[15] Wright, S. E. (2012). New linearizations of quadratic assignment problems. Computers & Operations Research, 39, 2858–2866.

[16] Loiola, E. M., Abreu, N., Boaventura-Netto, P. O., Hahn, P., & Querido, T. (2007). A survey for the quadratic assignment problem. European Journal of Operational Research, 176(2), 657–690.

[17] Wu, Z., Yang, Y., Bai, F., & Tian, J. (2012). Global optimality conditions and optimization methods for quadratic assignment problems. Applied Mathematics and Computation, 218, 6214–6231.

[18] Nyberg, A., & Westerlund, T. (2012). A new exact discrete linear reformulation of the quadratic assignment problem. European Journal of Operational Research, 220, 314–319.

[19] Zhang, H., Beltran-Royo, C., & Ma, L. (2013). Solving the quadratic assignment problem by means of general purpose mixed integer linear programming solvers. Annals of Operations Research, 207(1), 261–278.

[20] Kaufman, L., & Broeckx, F. (1978). An algorithm for the quadratic assignment problem using Benders' decomposition. European Journal of Operational Research, 2(3), 204–211.

[21] Adams, W. P., & Sherali, H. D. (1986). A Tight Linearization and an Algorithm for Zero-One Quadratic Programming Problems. Management Science, 32(10), 1274-1290.

[22] Adams, W. P., & Sherali, H. D. (1990). Linearization Strategies for a Class of Zero-One Mixed Integer Programming Problems. Operations Research, 38(2), 217-226.

[23] Adams, W., & Waddell, L. (2014). Linear programming insights into solvable cases of the quadratic assignment problem. Discrete Optimization, 14, 46–60.

[24] Tosun, U. (2015). On the performance of parallel hybrid algorithms for the solution of the quadratic assignment problem. Engineering Applications of Artificial Intelligence, 39, 267–278.

[25] Benlic, U., & Hao, J. K. (2015). Memetic search for the quadratic assignment problem. Expert Systems with Applications, 42, 584–595.

[26] Lawler, E. L. (1963). The Quadratic Assignment Problem. Management Science, 9(4), 586-599.

[27] Chen, D. S., Batson, R. G., & Dang, Y. (2010). Applied integer programming: Modeling and solution. USA: John Wiley & Sons.

[28] Nunkaew, W., & Phruksaphanrat, B. (2014). Lexicographic fuzzy multi-objective model for minimization of exceptional and void elements in manufacturing cell formation. International Journal of Production Research, 52(5), 1419–1442.

[29] Nunkaew, W., & Phruksaphanrat, B. (2015). Multi-Objective Programming Model for CNC Tools-Parts Clustering Based on the Theory of Group Technology. Paper presented at the The 3rd Annual Conference on Engineering and Information Technology (ACEAIT), Osaka, Japan.

[30] Nunkaew, W., & Phruksaphanrat, B. (2014). Novel Efficient Relaxation Reference Method for Part-Machine Clustering with Assignment of Duplicated Machines. Paper presented at the International Conference on Information Science, Electronics and Electrical Engineering (ISEEE), Sapporo, Japan.